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ENERGY ESTIMATES OF A SPECIAL FORM FOR SOLUTIONS OF THIRD-ORDER EQUATIONS OF THE PSEUDO-ELLIPTIC TYPE

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Abstract. *This article considers a boundary value problem for a third-order equation of the "pseudo-elliptic" type. Special energy estimates are established for the generalized solution of the equation. With the help of which you can build a solution to the boundary value problem in unlimited areas, in classes of functions growing at infinity, depending on the geometric characteristics of the boundaries of the area.*
Keywords*: third-order equation, bounded domains, unbounded domains, boundary value problem, energy estimates, Saint-Venant's principle, generalized solution, cutoff function.*

I. INTRODUCTION

This article explores the equations

$$L_0 lu + L_1 u + Mu = f(x,y,t) \qquad (1)$$

in an unrestricted area $Q = G\times(0,T)$, $G = D\times\Omega$. Where $D\subset R_x^n$ – bounded domain, $\Omega=\{y: y_1>0\}\subset R_y^m$ – unbounded domain, with smooth borders Γ_1 and Σ respectively,

$$lu = u_t + \alpha^k(x,y,t)u_{x_k} + \alpha(x,y,t)u,\;\; L_1u = b^{ij}(x,y,t)u_{x_ix_j} + b^i(x,y,t)u_{x_i},$$

$$L_0u = u_t - a^{ij}(x,y,t)u_{x_ix_j} + a^i(x,y,t)u_{x_i} + a(x,y,t)u,$$

$$Mu = c^{pq}(x,y,t)u_{y_py_q} + c^p(x,y,t)u_{y_p} + c(x,y,t)u.$$

Equation (1) was studied in the works [1] and [2]. In [1], a technique was developed for constructing regular solutions to boundary value problems for equation (1) in bounded domains, and the behavior of the solution was studied for $t\to\infty$.

Further, in the work [2], energy estimates of the type of the Saint-Venant's principle were established for generalized solutions of boundary value problems of equation (1). These estimates make it possible to determine the class of uniqueness of solutions to boundary value problems, in classes of functions growing at infinity depending on the geometric properties of the domain, and also allows us to study the properties of solutions to equation (1) in the neighborhood of irregular boundary points and infinitely distant points of the boundary of the domain.

II. LITERATURE ANALYSIS

It is known that in the study of properties and the construction of solutions to the problem of elasticity theory in unbounded domains, a special place is occupied by energy estimates of the type of the Saint-Venant's principle (see [3-7]). The study of this issue, both for the flat theory of elasticity and for the system of elasticity theory, was devoted to a series of works by O.A.Oleinik and her students, who managed to obtain accurate estimates that take into account the geometric characteristics of the area. In addition, they obtained uniqueness theorems and an existence theorem for solutions to boundary value problems in classes of functions growing at infinity depending on geometric characteristics. Based on these estimates, the behavior of the solution of the elasticity theory equations in the vicinity of irregular and infinitely distant points of the boundary was studied (see [8-11]). In these works, methods were developed for studying the behavior of solutions of elliptic and parabolic equations.

At present, due to the emergence of non-classical problems devoted to the study of large waves described by an odd-order equation, interest in studying the odd-order equation has received a new impetus [12-18]. Therefore, the construction of a solution to boundary value problems for odd-order equations, both in bounded and unbounded domains, is of particular interest. The study of the properties of these solutions in the neighborhood of irregular and infinitely distant points of the boundary occupies a separate place in the study of odd equations. The solution to these problems is the establishment of energy estimates of a special type. Therefore, the establishment of an analogue of the Saint-Venant's principle for solutions of an equation of an odd order and their application to the study of solving boundary value problems is one of the targeted scientific types of research in the field of mathematics.

The purpose of this work is to establish energy estimates of a special form, which allows constructing a solution to boundary value problems for equation (1) in unbounded domains, in classes of functions growing at infinity, depending on the geometric characteristics of the boundaries of the domain.

III. RESULTS
STATEMENT OF THE PROBLEM

Equation (1) will be considered with the following boundary conditions

$$u\big|_{\partial Q}=0,\quad \alpha^k u_{x_k}\Big|_{\sigma_2}=0. \qquad (2)$$

Where $\sigma_2=\{(x,y,t)\in\partial G\times(0,T):\alpha^k v_k<0\}$, v_k – internal normal vector to Q in the point (x,y,t).

Let us assume that the coefficients of operators L_0 and M satisfy the conditions

$$a^{ij}=a^{ji},\quad \lambda_0|\xi|^2\le a^{ij}\xi_i\xi_j\le\lambda_1|\xi|^2,\ (x,y,t)\in Q\cup\partial Q,\ \xi\in\mathbb{R}^{n+m+1},$$

$$c^{pq}=c^{qp},\quad \mu_0|\xi|^2\le c^{pq}\xi_p\xi_q\le\mu_1|\xi|^2,\ (x,y,t)\in Q\cup\partial Q,\ \xi\in\mathbb{R}^{n+m+1}.$$

We will assume that in some neighborhood of any of its points the hyperspace ∂G represent in the form

$$\begin{cases} x_j=\chi(x_1,x_2,\dots,x_{j-1},x_{j+1},\dots,x_n),\\ y_k=\chi_1(y_1,y_2,\dots,y_{k-1},y_{k+1},\dots,y_n)\end{cases},$$

where χ and χ_1 is a twice continuously differentiable function.

We make a division of the border of the region Q in the following way:

$$\sigma_1=\{(x,y,t)\in\partial G\times(0,T):\alpha^k v_k=0\},$$

$$\sigma_1=\{(x,y,t)\in\partial G\times(0,T):\alpha^k v_k>0\},$$

$$\sigma_2=\{(x,y,t)\in\partial G\times(0,T):\alpha^k v_k<0\}.$$

Let $\{Q_\tau\}=\{G_\tau\}\times(0,T)-$ a family of finite sub regions of a region Q, parameter dependent $\tau\in\Pi=\{\tau:0\le\tau\le\tau_0\}$, $\tau_0\le\infty$. We assume that $Q_\tau\subset Q_{\tau'}$ if $\tau\le\tau'$. We denote $S_\tau=s_\tau\times(0,T)=\partial Q_\tau\setminus\partial Q$. We assume that $s_\tau-$connected $(n+m-1)-$ dimensional surface having the same smoothness as ∂G, and its border $\partial s_\tau\subset\partial G$.

We assume for $\hat\tau\in\Pi$, $\hat\tau\ne0$ in some neighborhood $S_{\hat\tau}$ we can enter local coordinates $z_j=\varphi_j(x)$. This function $\varphi_j(x)$ is continuously differentiable.

We put $\Gamma_\tau=\partial G\cap\partial G_\tau$,

$$\sigma_{0,\tau}=\left\{x\in\Gamma_\tau\times(0,T):\alpha^k v_k=0\right\},$$
$$\sigma_{1,\tau}=\left\{x\in\Gamma_\tau\times(0,T):\alpha^k v_k>0\right\},$$
$$\sigma_{2,\tau}=\left\{x\in\Gamma_\tau\times(0,T):\alpha^k v_k<0\right\}.$$

For some $h>0$ define $\sigma_{2,h,\tau}=\left\{(x,y,t)\in\sigma_{2,\tau}:\rho((x,y,t),\partial\sigma_{2,\tau})>h\right\}$, $\sigma_{2,\tau}^h=\sigma_{2,\tau}\setminus\sigma_{2,h,\tau}$.

Let $E(Q_\tau)$ there are many functions $\upsilon\in C^2(Q_\tau)$ such that $\upsilon=0$ on the $\partial G_\tau\times(0,T)$, and for some number $h>0$ will be $\alpha^k\upsilon_{x_k}=0$ on the $\sigma_{0,\tau}\cup\sigma_{1,\tau}\cup\sigma_{2,\tau}^h$.

Let $H(Q_\tau)$ is the Hilbert space obtained by completion $E(Q_\tau)$ according to the norm

$$\|u\|_{H(Q_\tau)}=\left\{\int_{Q_\tau}\left(d_1^{ij}u_{x_i}u_{x_j}+u_{y_p}u_{y_q}+u_t^2+u^2\right)dx-\int_{\sigma_{2,\tau}}\alpha^k v_k a^{ij}u_{x_i}u_{x_j}ds\right\}^{\frac{1}{2}}.$$

Where

$$d_1^{ij}=-\frac{1}{2}\alpha^j a_{x_j}^{ij}-\frac{1}{2}a_t^{ij}+\alpha^j a^i+d^{ij}-\frac{1}{2\lambda_0}a^{ij},\quad d^{ij}=b^{ij}+\alpha^k a_{x_k}^{ij}-\alpha a^{ij}+a_t^{ij},$$
$$d^{ij}=d^{ji},\quad \gamma_0|\xi|^2\le d^{ij}\xi_i\xi_j\le\gamma_1|\xi|^2,\ (x,y,t)\in Q\cup\partial Q,\ \xi\in\mathbb{R}^{n+m+1},$$
$$d_1^{ij}=d_1^{ji},\quad \beta_0|\xi|^2\le d_1^{ij}\xi_p\xi_q\le\beta_1|\xi|^2,\ (x,y,t)\in Q\cup\partial Q,\ \xi\in\mathbb{R}^{n+m+1}.$$

Consider the bilinear form

$$a(u,\upsilon)=\int_{Q_\tau}\left(\alpha^k a^{ij}u_{x_i}\upsilon_{x_jx_k}+a^{ij}u_{x_i}\upsilon_{x_jt}+\left(\alpha^k a_{x_j}^{ij}-\alpha^i a^k\right)u_{x_i}\upsilon_{x_j}+d^{ij}u_{x_i}\upsilon_{x_j}\right)dxdydt+$$
$$\int_{Q_\tau}\left(\left(d^i-d_{x_j}^{ij}\right)u\upsilon_{x_i}+\left(a_{x_i}^{ij}+a^i+\alpha^i\right)u_{x_i}\upsilon_t+c^{pq}u_{y_p}\upsilon_{y_q}+\left(c^p-c_{y_q}^{pq}\right)u_{y_p}\upsilon\right)dxdydt+$$
$$\int_{Q_\tau}\left(u_t\upsilon_t+(\alpha+a)u\upsilon_t+\left(c_{y_p}^p-c-c_{y_py_q}^{pq}+d+d_{x_i}^i+d_{x_ix_j}^{ij}\right)u\upsilon\right)dxdydt.$$

where $d^i=b^i+\alpha a^i-\alpha^i a_{x_k}^k+\alpha^i a-a_t^i$.

Definition. Function $u(x,y,t)$ will be called a generalized solution of problem (1), (2) in the domain Q, if for any finite sub-domain Q_τ domain Q we have $u(x)\in H(Q_\tau)$ and the relation

$$a(u,\upsilon)=\int_{Q_\tau}\upsilon f(x,y,t)dxdydt \qquad (3)$$

For an arbitrary function $\upsilon(x,y,t)\in E(Q_\tau)$ satisfying the condition $\upsilon=0$ on the S_τ.

Let $S_\tau=\Omega\cap\{x:x_1=\tau+\gamma\}$, $\gamma=const$ for some $0\le\tau\le\tau_0$.

We enter the designation $E(u)=d^{ij}u_{x_i}u_{x_j}+c^{pq}u_{y_p}u_{y_q}+u_t^2-\theta u^2$,

$$B(x,y,t)=\max\left\{2^{-1}\left(c^p-c^{pq}_{y_p}\right),0\right\},\ P(\tau)=\sup_{S_\tau}B(x,y,t),\ g(\tau)=\sup_{S_\tau}\left|c^{pq}v_pv_q\right|^{\frac{1}{2}},$$

$$0<\lambda(\tau)\le\inf_{\upsilon\in N}\left\{\int_{S_\tau}E(\upsilon)ds\left|\int_{S_\tau}\upsilon^2ds\right|^{-1}\right\}, \qquad (4)$$

where $N-$many functions $\upsilon(x)$ continuously differentiable in a neighborhood S_τ, at $(x,y,t)\in\bar{Q}$, which are equal to zero on $S_\tau\cap\Gamma$.

We assume that there exists such a positive, absolutely continuous function $\Phi(\tau)$, $\tau\in\Pi$, such that it will be determined by the inequalities

$$\Phi(\tau)\ge g(\tau)\lambda^{-\frac{1}{2}}(\tau)+P(\tau)\lambda^{-1}(\tau). \qquad (5)$$

Function $\tau(\alpha)$ is a solution of the problem

$$\frac{d\tau}{d\alpha}=\Phi(\tau),\quad \tau(0)=0. \qquad (6)$$

Theorem 1. (An analogue of Saint-Venant's principle) Let $-1\le a^{ij}_{x_i}+a^i+a\le 0$; $\theta\equiv d_0-\frac{1}{2}d^{ij}_{x_ix_j}+\frac{1}{2}d^i_{x_i}-\frac{1}{2}c^{pq}_{y_py_q}+\frac{1}{2}c^p_{y_p}-c<0$. If function $u(x,y,t)$ is a generalized solution of equation (1) satisfying the boundary condition (2) in the domain Q, and $f(x,y,t)=0$ in the Q_{τ^0}. Then for any $0\le R_0\le R$ there is an estimate

$$\int_{Q_{\tau(R_0)}}E(u)dxdydt\le\exp\left[-(R-R_0)\right]\int_{Q_{\tau(R)}}E(u)dxdydt \text{ in the } Q_{\tau^0}. \qquad (4)$$

Proof. Let $R_0>0$. Then $\tau(R_0)>0$. For $\hat{\tau}\in\Pi$ such that $\tau(R_0)\le\hat{\tau}\le\tau(R)$, $(x,y,t)\in Q_{\tau(R)}$, construct a cutting function $\psi_\delta(y,\hat{\tau})$, parameter dependent δ, where $0<2\delta<\min_l\hat{\tau}_l$. To this end, consider the function

$$g_\delta(\xi,\eta)=\int_{-\infty}^{\xi-\delta}\delta^{-1}\omega\left(\frac{\eta-\theta}{\delta}\right)d\theta,\ \theta\in(-\infty,\infty). \qquad (5)$$

where $\omega(\theta)-$ an infinitely differentiable function such that $\omega(\theta)=0$ at $|\theta|\ge1$, $\omega(\theta)>0$ at $|\theta|<1$, $\omega(\theta)\le1$ and $\int_{-\infty}^{\infty}\omega(\theta)d\theta=1$.

It is obvious that

$$g_\delta(\xi,\eta)=\begin{cases}1, & \eta\le\xi-2\delta,\\ 0, & \eta\ge\xi\end{cases},$$

$$0\le g_\delta(\xi,\eta)\le1.$$

We put $\psi_\delta(y,\hat{\tau})=1$, if $y\in\bar{Q}_0$. Every point $y\in\bar{Q}_{\tau(R)}\setminus\bar{Q}_0$ matches a single value $\tau=\tau(y)$ such that $y\in S_\tau$.

We put $\psi_\delta(y,\hat{\tau})=g_\delta(\hat{\tau},\tau(y))$ for $y\in\bar{Q}_{\tau(R)}\setminus\bar{Q}_0$. Here it is obvious that

$$\psi_\delta(y,\hat{\tau})=\begin{cases}1, & y\in\bar{S}_\tau,\quad \tau\le\hat{\tau}-2\delta,\\ 0, & y\in\bar{S}_\tau,\quad \tau\ge\hat{\tau}\end{cases}$$

By virtue of the assumptions, the function $\varphi_\delta(y,\hat{\tau})$ is a continuously differentiable function with respect to the variables y and parameter $\hat{\tau}$.

Let $u_m(x,y,t)$ – sequence of functions form $E(Q_\tau)$ such that $u_m(x,y,t)\to u(x,y,t)$ at $m\to\infty$ in $H(Q_\tau)$.

We set in identity (3) $Q_\tau = Q_{\hat{\tau}}$, $\upsilon = u_m(x,y,t)\psi_\delta(y,\hat{\tau})$. Further, integrating by parts, we have

$$\int\limits_{Q_{\hat{\tau}}} E(u)dxdydt \le E_{m,\delta}(\hat{\tau}) + \varepsilon_{m,\delta}(\hat{\tau}) + \int\limits_{S_{\hat{\tau}}} c^{pq} u_{my_p} u_m v_q ds + \frac{1}{2}\int\limits_{S_{\hat{\tau}}} \left(c^p - c^{pq}_{y_q}\right) u_m^2 v_p ds . \qquad (6)$$

Now, using the Cauchy-Bunyakovsky inequality, from (6) we obtain

$$\int\limits_{Q_{\hat{\tau}}} E(u)dxdydt \le \left|E_{m,\delta}(\hat{\tau})\right| + \left|\varepsilon_{m,\delta}(\hat{\tau})\right| + \Phi(\hat{\tau})\int\limits_{S_{\hat{\tau}}} E(u)dxdydt \ . \qquad (7)$$

We put $F(\hat{\tau}) = \int\limits_{Q_{\hat{\tau}}} E(u)dxdydt$. Because $\dfrac{\partial F(\hat{\tau})}{\partial \hat{\tau}} = \int\limits_{S_{\hat{\tau}}} E(u)ds$, then from inequality (7) we obtain that

$$F(\hat{\tau}) \le \left|E_{m,\delta}(\hat{\tau})\right| + \left|\varepsilon_{m,\delta}(\hat{\tau})\right| + \Phi(\hat{\tau})\frac{\partial F(\hat{\tau})}{\partial \hat{\tau}} \ . \qquad (8)$$

Here believing $\hat{\tau} = \tau(\alpha)$, $R_0 \le \alpha \le R$, we get

$$F(\tau(\alpha)) \le \left|E_{m,\delta}(\tau(\alpha))\right| + \left|\varepsilon_{m,\delta}(\tau(\alpha))\right| + \frac{dF(\tau(\alpha))}{d\alpha}.$$

Now multiplying this inequality by $e^{-\alpha}$ and integrating it over α we have

$$F(\tau(R_0)) \le \exp\left(-R + R_0\right)F(\tau(R)) + e^{R_0}\int\limits_{R_0}^{R}\left(\left|E_{m,\delta}(\tau(\alpha))\right| + \left|\varepsilon_{m,\delta}(\tau(\alpha))\right|\right)e^{-\alpha}d\alpha . \qquad (9)$$

Here the second term on the right side of inequality (9) is arbitrarily small if m big enough and $\delta = \delta(m)$ a little enough.

In inequalities (9), passing to the limit $m\to\infty$ и $\delta(m)\to 0$, we obtain estimate (4).

Applying inequalities (4), one can obtain a uniqueness theorem and an existence theorem, a solution to boundary value problems for equations (1) in the classes of functions of domains growing at infinity, depending on the behavior of the coefficients of the equation and depending on the geometric characteristics of the domain.

Here, as an application of inequality (4), we present the uniqueness theorem for the solution.

Theorem 2. Let $u(x,y,t)$ is a generalized solution of problem (1), (2) in an unbounded domain Q, and $f(x,y,t)=0$ in Q. Let a family of finite sub-domains be defined Q_τ domain Q, and $Q = \bigcup\limits_{\tau\in\Pi} Q_\tau$, $\Pi = \{\tau : 0 \le \tau < \infty\}$, $\tau(\alpha)\to\infty$ at $\alpha\to\infty$, $\dfrac{d\tau}{d\alpha} = \Phi(\tau)$, $\tau(0)=0$. Then if for some sequence of real numbers $\{R_j\}$ such that $R_j\to\infty$ at $j\to\infty$, the inequalities

$$\int\limits_{Q_{\tau(R_j)}} E(u)dxdydt \le \varepsilon(R_j)\exp\{R_j\}, \ \ j = 1,2,3,4,... \qquad (10)$$

where $\varepsilon(R_j)\to 0$ at $j\to\infty$, then $u\equiv 0$ in Q.

Proof. We fix an arbitrary number R_k from sequence $\{R_j\}$. Then from (4) follows

$$\int_{Q_{\tau(R_k)}} E(u)dxdydt \le \exp\left[-(R_{k+s}-R_k)\right] \int_{Q_{\tau(R_{r+s})}} E(u)dxdydt .$$

Therefore, by virtue of condition (10), we have that

$$\int_{Q_{\tau(R_k)}} E(u)dxdydt \le \varepsilon(R_{k+s})\exp\{R_k\} .$$

From here at $s \to \infty$, we get $u \equiv 0$ in $Q_{\tau(R_k)}$. Then k chosen arbitrarily then $u \equiv 0$ in Q.

IV. REFERENCES

1. Kozhanov A. I. Boundary Value Problems for Equations of Mathematical Physics of Odd Order. – Novosibirsk, 1990. – 130 p.
2. Khashimov A. R., Smetanova D. On the Uniqueness Classes of Solutions of Boundary Value Problems for Third-Order Equations of the Pseudo-Elliptic Type // Axioms. – 2020. – Vol. 9. – P. 80.
3. Flavin J. N. On Knowles version of Saint-Venant's principle in the two dimensional electrostatics // Arch. Rational Mech. And Anal. – 1974. – Vol. 53. – P. 53-62.
4. Marin M., Oechsner A., Craciun E. M. A generalization of the Saint-Venant's principle for an elastic body with dipolar structure // Continuum Mechanics and Thermodynamics. – 2020. – Vol. 32(1). – P. 269-278.
5. Xiao B., Sun Z., Shi S. On random elastic constraint conditions of Levinson beam model considering the violation of Saint-Venant's principle in dynamic // European Physical Journal Plus. – 2020. – Vol. 135(2). – P. 168.
6. Todorcˇevic´ V. Subharmonic behavior and quasi-conformal mappings // Anal. Math. Phys. – 2019. – Vol. 9. – P. 1211-1225.
7. Roohi M., Soleymani K., Salimi M. Numerical evaluation of the general flow hydraulics and estimation of the river plain by solving the Saint-Venant equation // Modeling Earth Systems and Environment. – 2020. – Vol. 6(2). – P. 645-658.
8. Oleinik O. A., Iosifyan G. A. A priori estimates of solutions of the first boundary value problem for the system of equations of the theory of elasticity and their applications // Uspekhi Mat. Nauk. – 1979. – Vol. 32(5). – P. 193.
9. Oleinik O. A., Iosifyan G. A. On the principle of Saint-Venant in the plane theory of elasticity // DAN USSR. – 1978. – Vol. 239(3). – P. 530.
10. Oleinik O. A., Iosifyan G. A. The Saint-Venant principle in the plane theory of elasticity and boundary value problems for the biharmonic equation in unbounded domains // Siberian Mathematical Journal. – 1978. – Vol. 19(5). – P. 1154.
11. Oleinik O. A. On the behavior of solutions of linear parabolic systems of differential equations in unbounded domains // Advances in Mathematical Sciences. – 1975. – Vol. 30(2). – P. 219-220.
12. Fabiano N., Nikolic´ N., Shanmugam T. et al. Tenth order boundary value problem solution existence by fixed point theorem // J. Inequal Appl. – 2020. – P. 166.

13. Faminskii A. V., Larkin N. A. Odd-order quasi-linear evolution equations posed on a bounded interval // Bol. Spc. Paran. Mat. (3s.). – 2010. – Vol. 28(1). – P. 67-77.

14. Freire I. L., Sampaio J. C. S. Nonlinear self-adroitness of a generalized fifth-order KdV equation // J. Phys. A: Math. Theor. – 2012. – Vol. 45. – P. 032001 (7pp).

15. Shanmugam T., Muthiah M., Radenovic´ S. Existence of Positive Solution for the Eighth-Order Boundary Value Problem Using Classical Version of Leray–Schauder Alternative Fixed-Point Theorem // Axioms. – 2019. – Vol. 8. – P. 129.

16. Khashimov A., Yakubov S. On some properties of solutions of the Cauchy problem for a non-stationary equation of the third order of composite type // Ufa Mathematical Journal. – 2014. – Vol. 6(4). – P. 139-148.

17. Khashimov A. R., Smetanova D. Nonlocal Problem for a Third-Order Equation with Multiple Characteristics with General Boundary Conditions // Axioms. – 2021. – Vol. 10. – P. 110.

18. Khashimov A. R. On the uniqueness of solutions of the second boundary value problem for a third order equation of composite type in unbounded domains // Bulletin of the Samara Technical University, Series of Physical and Mathematical Sciences. – 2012. – No. 2(27). – P. 18-25.