ON THE UNIQUENESS OF THE SOLUTION OF A NONLINEAR BOUNDARY VALUE PROBLEM FOR A THIRD-ORDER EQUATION WITH MULTIPLE CHARACTERISTICS ON A PLANE

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Abstract. *In this paper, we study a nonlinear boundary value problem for a third-order partial differential equation with multiple characteristics defined on a plane domain. Equations of this type arise in various problems of mathematical physics and often present significant analytical difficulties due to the presence of nonlinear terms associated with multiple characteristic directions.*

We formulate an appropriate boundary value problem and analyze the conditions under which the problem admits a solution in the class of continuous functions. Particular attention is devoted to the influence of the nonlinear structure and the characteristic properties of the differential operator on the solvability of the problem. The main result of the paper establishes the uniqueness of a solution in the specified functional class. The proof is based on the construction of suitable energy integrals and the application of the energy integral method. The obtained results contribute to the theory of nonlinear boundary value problems for higher-order partial differential equations with multiple characteristics. They may be useful in the further study of similar classes of nonlinear equations.

Keywords*: a nonlinear boundary value problem, a solution, a uniqueness, the energy integral method .*

INTRODUCTION

In a line the following equation refers to poorly studied odd-order equations

$$L(u) \equiv u_{xxx} - u_t = f(x, y, u, u_x, u_{xx}), \quad u = u(x,t), \qquad (MC_1)$$

which is called an equation with multiple characteristics (*MC*). (see[2])

The equation with multiple characteristics (MC) arises in various problems of physics and mechanics, making it of significant theoretical and applied interest.

The well-known Korteweg-de Vries equation *(KdV)*

$$u_y + uu_x + \beta u_{xxx} = 0 \qquad (KdV)$$

which is the object of research by many authors and occupies an important place in the study of nonlinear wave propagation in weakly dispersive media [3 – 7].

The Korteweg–de Vries (*KdV*) equation describes the evolution of weakly nonlinear long-wavelength excitations in a medium with dispersion in the high-frequency domain. The KdV equation finds applications in diverse fields, such as fluid dynamics (e.g., modeling gravitational waves in shallow water and nonlinear Rossby waves), plasma physics (e.g., describing ion-acoustic waves), electrical engineering (e.g., analyzing nonlinear circuits), and even epidemiology (e.g., simulating the time evolution of infected individuals during an epidemic), etc. [3 – 7].

LITERATURE REVIEW

Reference [2] established a foundational framework for the equation,

$$u_{xxx} - u_t = 0, \qquad (LKdV)$$

including the construction of a fundamental solution, the development of potential theory, and a methodology for addressing boundary value and Cauchy problems.

The solutions of equation

$$u_{xxx} + u_{yyy} - u_t = 0 \qquad (MC_2)$$

and the linear Zakharov-Kuznetsov equation (equations, as referenced in [12 - 14])

$$u_t + u_{xxx} + u_{yyx} = 0 \qquad (LZK)$$

exhibit similar asymptotic behavior at infinity.

As a multidimensional generalization of the Korteweg-de-Vries equation, the Zakharov-Kuznetsov equation (*LZK*) describes ion-acoustic wave propagation in plasma [14].

A fundamental solution for equation (MC_2) within the space R^{n+1} was constructed in [1].

The first paper addressing the initial-boundary value problem of the Korteweg-de Vries (KdV) equation on the finite interval (0,1) was by Bubnov in 1979 [8], who considered the initial-boundary value problem with general linear boundary conditions. Since then, numerous authors have worked on improving existing results and presenting new findings in recent years [12 – 13].

The initial-boundary value problem of the Zakharov-Kuznetsov equation was studied in [15 – 18].

The relevance of studying boundary value problems for odd-order equations with multiple characteristics is underscored by these and other practical applications. It's worth noting that certain linear boundary value problems for linear third-order equations with multiple characteristics have been explored in [2, 9 - 11] in a line.

This paper employs a linear boundary value problem for a linear third-order equation with multiple characteristics in a plane.

METHODOLOGY

To establish the uniqueness of the solution to the considered nonlinear boundary value problem for a third-order equation with multiple characteristics, the energy integral method is employed. This approach is widely used in the analysis of partial differential equations because it allows one to derive a priori estimates that play a crucial role in proving uniqueness and stability of solutions.

First, an appropriate energy functional associated with the differential equation is constructed. This functional is obtained by multiplying the equation by a suitable auxiliary function related to the solution and integrating over the considered domain. The resulting energy integral represents a quantitative measure of the behavior of the solution within the domain.

Next, several elementary inequalities are applied to estimate the terms appearing in the energy integral. These inequalities help control nonlinear components of the equation and ensure that the integral expressions remain bounded under the imposed boundary conditions. In addition, inequalities of the Friedrichs type are used to relate the norms of the solution and its derivatives within the domain. Such inequalities are essential in deriving estimates that connect the behavior of the solution in the interior of the domain with its boundary values.

Using these estimates, it is shown that if two solutions of the boundary value problem exist, then their difference satisfies an energy inequality that forces the corresponding energy functional to vanish. This implies that the difference between the solutions is identically zero throughout the domain.

Consequently, the obtained estimates guarantee that the boundary value problem admits at most one solution in the considered class of continuous functions, thereby establishing the uniqueness of the solution.

ANALYSIS AND RESULTS

Problem. It is required to determine in domain $D=\{(x,y,t): 0<x<1,\ 0<y<1,\ 0<t\le T\}$ function u (x, y, t) that has the following properties:

1) $u(x,y,t)\in C^{3,3,1}_{x,y,t}(D)\cap C^{2,2,0}_{x,y,t}(\overline{D})$;

2) which is a regular solution to the following equation:

$$L(u)\equiv u_{xxx}+u_{yyy}-u_t=f(x,y,t) \qquad (1)$$

in domain D;

3) satisfying the following conditions

$$u(x,y,0)=u_0(x,y),\quad 0\le x\le 1,\ 0\le y\le 1, \qquad (2)$$

$$u_{xx}(0,y,t)=g_1(u(0,y,t)),\quad 0\le y\le 1, 0\le t\le T, \qquad (3)$$

$$u_x(0,y,t)=\varphi_1(y,t),\quad 0\le y<1, 0\le t\le T, \qquad (4)$$

$$u(1,y,t)=\varphi_2(y,t),\quad 0\le y\le 1, 0\le t\le T, \qquad (5)$$

$$u_{yy}(x,0,t)=g_2(u(x,0,t)),\quad 0\le x\le 1,\ 0\le t\le T, \qquad (6)$$

$$u_y(x,0,t)=\psi_1(t,x),\quad 0\le x\le 1,\ 0\le t\le T, \qquad (7)$$

$$u(x,1,t)=\psi_2(x,t),\quad 0\le x\le 1, 0\le t\le T, \qquad (8)$$

and matching conditions

$$\frac{\partial^2 u_0(0,0)}{\partial x^2}=\frac{\partial^2 u_0(0,0)}{\partial y^2}=g_1(u(0,0,0))=g_2(u(0,0,0)),$$

$$\frac{\partial u_0(0,0)}{\partial x}=\varphi_1(0,0),\ \frac{\partial u_0(0,0)}{\partial y}=\psi_1(0,0),\qquad u_0(1,0)=\varphi_2(0,0),\ u_0(0,1)=\psi_0(0,0).$$

Theorem (Uniqueness of solution). Let for $g_1(\mu, y, t), g_2(x, \nu, t)$ be continuous functions of their arguments $0 \le x \le 1, 0 \le y \le 1, 0 \le t \le T$ for any $|\mu| < \infty, |\nu| < \infty$ satisfying the following condition

$$|g_1(\mu_1, y, t) - g_1(\mu_2, y, t)| \le l_1 |\mu_1 - \mu_2|, \qquad (6)$$

$$|g_2(x, \nu_1, t) - g_2(x, \nu_2, t)| \le l_2 |\nu_1 - \nu_2|, \qquad (7)$$

$$0 < l_1 \le \frac{1}{2}, \qquad (8)$$

$$0 < l_2 \le \frac{1}{2}, \qquad (9)$$

where

$\mu = \mu(y, t) = u(0, y, t),\ \nu = \nu(x, t) = u(x, 0, t),$

and $l_1 > 0$, $l_2 > 0$, $L > 0$ some constants.

Then the solution to Problem A is unique.

Proof. Let there be two solutions to the considered problem, u_1 and u_2. Consider their difference $w = u_1 - u_2$. With regard to w, we obtain the following problem:

$$L(w) \equiv w_{xxx} + w_{yyy} - w_t = 0 \qquad (1_0)$$

$$w(x, y, 0) = 0,\ 0 \le x \le 1, \qquad (2_0)$$

$$w_{xx}(0, y, t) = g_1(u_1(0, y, t)) - g_1(u_2(0, y, t)),\quad 0 \le y \le 1, 0 \le t \le T, \qquad (3_0)$$

$$w_x(0, y, t) = 0, 0 \le y \le 1,\ 0 \le t \le T, \qquad (4_0)$$

$$w(1, y, t) = 0,\ 0 \le y \le 1, 0 \le t \le T, \qquad (5_0)$$

$$w_{yy}(0, y, t) = g_2(u_1(x, 0, t)) - g_2(u_2(x, 0, t)),\quad 0 \le x \le 1, 0 \le t \le T, \qquad (6_0)$$

$$w_y(x, 0, t) = 0,\quad 0 \le x \le 1,\ 0 \le t \le T, \qquad (7_0)$$

$$w(x, 1, t) = 0,\quad 0 \le x \le 1, 0 \le t \le T. \qquad (8_0)$$

Let us prove that $w(x, y) \equiv 0$.

Having integrated the following identity

$$\sigma(x, y, t) w(x, y, t) L(w) \equiv \sigma w(w_{xxx} + w_{yyy} - w_t) = 0 \qquad (10)$$

$\sigma = \sigma(x, y, t) = e^{-x-y-3t}$ over domain D,

$$\int_0^T \int_0^1 \sigma w w_{xx} \Big|_{x=0}^{x=1} dydt - \int_0^T \int_0^1 \sigma_x w w_x \Big|_{x=0}^{x=1} dydt - \frac{1}{2}\int_0^T \int_0^1 \sigma w_x^2 \Big|_{x=0}^{x=1} dydt + \frac{1}{2}\int_0^T \int_0^1 \sigma_{xx} w^2 \Big|_{x=0}^{x=1} dydt + \frac{3}{2}\iiint_D \sigma_x w_x^2 dxdydt -$$

$$- \frac{1}{2}\iiint_D \sigma_{xxx} w^2 dxdydt + \int_0^T \int_0^1 \sigma w w_{yy} \Big|_{y=0}^{y=1} dxdt - \int_0^T \int_0^1 \sigma_y w w_y \Big|_{y=0}^{y=1} dxdt - \frac{1}{2}\int_0^T \int_0^1 \sigma w_y^2 \Big|_{y=0}^{y=1} dxdt + \frac{1}{2}\int_0^T \int_0^1 \sigma_{yy} w^2 \Big|_{y=0}^{y=1} dxdt +$$

$$+ \frac{3}{2}\iiint_D \sigma_y w_y^2 dxdydt - \frac{1}{2}\iiint_D \sigma_{yyy} w^2 dxdydt - \frac{1}{2}\int_0^1 \int_0^1 \sigma w^2 \Big|_{t=0}^{t=T} dxdy + \frac{1}{2}\iint_D \sigma_t w^2 dxdy = 0,$$

taking into account boundary conditions (2_0) - (8_0),

$$-\int_0^T\int_0^1 (g_1(u_1)-g_1(u_2))\sigma w\big|_{x=0}dydt-\frac{1}{2}\int_0^T\int_0^1 \sigma w_x^2\Big|^{x=1}dydt-\frac{1}{2}\int_0^T\int_0^1 \sigma w^2\big|_{x=0}dydt-$$

$$-\frac{3}{2}\iiint_D \sigma w_x^2 dxdydt-\int_0^T\int_0^1 (g_2(u_1)-g_2(u_2))\sigma w\big|_{y=0}dxdt-\frac{1}{2}\int_0^T\int_0^1 \sigma w_y^2\Big|^{y=1}dxdt-$$

$$-\frac{1}{2}\int_0^T\int_0^1 \sigma w^2\big|_{y=0}dxdt-\frac{3}{2}\iiint_D \sigma w_y^2 dxdydt+$$

$$+\frac{1}{2}\iiint_D (\sigma_t-\sigma_{xxx}-\sigma_{yyy})w^2 dxdydt-\frac{1}{2}\int_0^1\int_0^1 \sigma w^2\Big|^{t=T}dxdy=0. \quad (11)$$

we introduce the following notation:

$$I=\frac{1}{2}\int_0^T\int_0^1 \sigma w_x^2\Big|_{x=1}dydt+\frac{1}{2}\int_0^T\int_0^1 \sigma w_y^2\Big|_{y=1}dxdt+\frac{1}{2}\int_0^1\int_0^1 \sigma w^2\Big|_{t=T}dxdy+$$

$$+\frac{3}{2}\iiint_D \sigma w_x^2 dxdydt+\frac{3}{2}\iiint_D \sigma w_y^2 dxdydt\geq 0 \quad (12)$$

According to notation (12), alternatively we have

$$I=-\int_0^T\int_0^1 (g_1(u_1)-g_1(u_2))\sigma w\big|_{x=0}dydt-\frac{1}{2}\int_0^T\int_0^1 \sigma w^2\big|_{x=0}dydt-$$

$$-\int_0^T\int_0^1 (g_2(u_1)-g_2(u_2))\sigma w\big|_{y=0}dxdt-\frac{1}{2}\int_0^T\int_0^1 \sigma w^2\big|_{y=0}dxdt+\frac{1}{2}\iiint_D (\sigma_t-\sigma_{xxx}-\sigma_{yyy})w^2 dxdy$$

(13)

With conditions (6) - (7), we have

$$I\leq\int_0^T\int_0^1 \left(l_1-\frac{1}{2}\right)w^2 v\Big|_{x=0}dydt+\int_0^T\int_0^1 \left(l_2-\frac{1}{2}\right)w^2 v\Big|_{y=0}dxdt-\frac{1}{2}\iiint_D w^2 v dxdy. \quad (14)$$

When conditions (8) and (9) are satisfied, from (14) we arrive at the following inequality $I\leq 0$. Hence, $I=0$.

Then from (12) we obtain the following conditions:

$$\begin{cases} w_x(1,y,t)=0\ ,\ 0\leq y\leq 1,\ 0\leq t\leq T,\\ w_y(x,1,t)=0\ ,\ 0\leq x\leq 1,\ 0\leq t\leq T,\\ w(x,y,T)=0,\ \ 0\leq x\leq 1,\ \ 0\leq y\leq 1,\\ w_x(x,y,t)=0,\ \ (x,y,t)\in D,\\ w_y(x,y,t)=0,\ \ (x,y,t)\in D\ . \end{cases}$$

Hence, we have

$w(x, y, t) = p(y, t),\ (x, y, t) \in D.$

Since $w(1, y, t) = 0$, $0 \le y \le 1, 0 \le t \le T$, , then $p(y, t) \equiv 0.$

Due to continuity $w(x, y, t)$ in $\overline{D}$ we have $w(x, y, t) = 0$.

CONCLUSION

In this paper, a nonlinear boundary value problem for a third-order partial differential equation with multiple characteristics defined on a plane domain has been investigated. The main objective of the study was to analyze the solvability of the problem and to establish the uniqueness of its solution within an appropriate functional framework.

By applying the energy integral method, together with elementary inequalities and Friedrichs-type inequalities, we established the uniqueness of the solution.

Based on these results, it has been proven that when the conditions of the uniqueness theorem are satisfied, the nonlinear boundary value problem under consideration admits a unique solution in the class of continuous functions. In particular, it was shown that if two solutions satisfy the same equation and boundary conditions, then their difference must vanish identically in the domain, which guarantees the uniqueness of the solution.

The obtained results contribute to the theoretical study of nonlinear boundary value problem for partial differential equations of higher order with multiple characteristics. The developed approach and the use of energy estimates may also be applied to the investigation of other classes of boundary value problems with similar structural properties.

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